International Journal of Theoretical Physics, Vol. 31, No. 11, 1992

Time-Dependent Variational Method for Sine-Gordon Quantum Field Theory

Anjana Sinha[1] and Rajkumar Roychoudhury[1]

Received October 25, 1991

The sine-Gordon model in 1 + 1 dimensions is studied within the Schrödinger framework for field theory. In particular we evaluate the effective potential and examine the finiteness of $m(t)$, the soliton mass, for all t.

1. INTRODUCTION

In dealing with time-dependent field-theoretic problems, the Schrödinger picture is particularly suitable (Jackiw and Kerman, 1979; Cooper and Mottola, 1987; Pi and Samiullah, 1987) as it gives a clear picture of the system's time evolution. The method is even better than the loop expansion, as it is nonperturbative in $\hbar$. The method has been applied to various quantum mechanical problems and the approximate results have been found to be in good agreement with the exact ones obtained numerically (Cooper *et al.*, 1986). Also, this method has been successfully applied in curved space Kim *et al.*, 1988; Roy, 1991).

In the present paper we study the sine-Gordon model in (1 + 1)-dimensional flat space-time and derive an expression for the soliton mass $m(t)$, where the time dependence is explicitly shown.

The organization of the paper is as follows. In Section 2, we give an outline of the method of calculating the effective potential in the Schrödinger picture. In Section 3, renormalization in the static case is discussed; the time dependence of G is evaluated in Section 4 for the free theory solution. In Section 5, the time dependence of $m^2(t)$ is explicitly calculated, keeping only next to leading order terms. Finally, Section 6 gives a summary and remarks.

[1]Indian Statistical Institute, Calcutta 700035, India.

0020-7748/92/1100-1961$06.50/0

2. FIELD THEORY IN THE SCHRÖDINGER PICTURE

In the functional Schrödinger picture (Pi and Samiullah, 1987; Cooper *et al.*, 1986; Kim *et al.*, 1988) an abstract quantum mechanical state $|\psi(t)\rangle$ is replaced by a wave functional $\Psi(\varphi, t)$, which is a functional of a *c*-number field $\varphi(x)$ at a fixed time t. We shall take our trial wave functional to be Gaussian, centered at φ, with width given by G.

To be precise, we take

$$\Psi(\varphi, t) = N \exp\left[-\int_{x,y} \bar{\varphi}(x) B(x,y) \bar{\varphi}(y) + \frac{i}{\hbar}\int_x \hat{\pi}(x)\bar{\varphi}(x) - i\int_{x,y} G(x,y)\Sigma(x,y)\right] \tag{1}$$

where N is the normalization factor and

$$\bar{\varphi}(x) \equiv \varphi(x) - \hat{\varphi}(x,t) \tag{2}$$

$$B(x,y) \equiv \frac{1}{4\hbar} G^{-1}(x,y,t) - \frac{i}{\hbar}\Sigma(x,y,t) \tag{3}$$

The following expectation values are obtained easily by functional integration:

$$\langle \varphi(x) \rangle = \hat{\varphi}(x,t) \tag{4}$$

$$\left\langle -i\hbar \frac{\delta}{\delta\varphi(x)} \right\rangle = \hat{\pi}(x,t) \tag{5}$$

$$\langle \varphi(x)\varphi(y) \rangle = \hat{\varphi}(x,t)\hat{\varphi}(y,t) + \hbar G(x,y,t) \tag{6}$$

$$\left\langle i\hbar \frac{\partial}{\partial t} \right\rangle = \int_x \hat{\pi}(x,t)\varphi(x,t) + \hbar\int_{x,y} \Sigma(x,y,t)\dot{G}(y,x,t) \tag{7}$$

$\hat{\varphi}$, $\hat{\pi}$, G, and Σ are the variational parameters. $\hat{\pi}$ and Σ play the role of conjugate momenta of φ and G, respectively.

The effective action in this picture is given by

$$\Gamma = \int dt \langle \psi(t) | i\hbar\partial_t - H | \psi(t) \rangle \tag{8}$$

where the Hamiltonian is given by

$$H \equiv \int_x \left[-\frac{\hbar^2}{2} \frac{\delta^2}{\delta\varphi^2(x)} + \frac{1}{2} (\nabla\varphi)^2 + V(\varphi) \right] \tag{9}$$

The sine-Gordon model in (1 + 1) dimensions is governed by the potential (Rajaraman, 1982)

$$V(\varphi) = \frac{\mu^4}{\lambda} \left(1 - \cos \frac{\sqrt{\lambda}}{\mu} \varphi \right) \tag{10}$$

The Euler–Lagrange equation for φ for the potential given in (10) has the static localized solution

$$\bar{\varphi}(x) = 4 \tan^{-1}[\exp(\mu x - \mu x_0)] \tag{11}$$

where

$$\bar{\varphi}(x) = \frac{\sqrt{\lambda}}{\mu} \varphi(x) \tag{12}$$

An outline of the solution (11) is sketched in Figure 1.

Now taking the expectation value with respect to the trial wave function (1), we get

$$\langle V(\varphi) \rangle = \frac{\mu^4}{\lambda} \left[1 - \cos\left(\frac{\sqrt{\lambda}}{\mu} \varphi \right) \exp\left(-\frac{\lambda \hbar G}{2\mu^2} \right) \right] \tag{13}$$

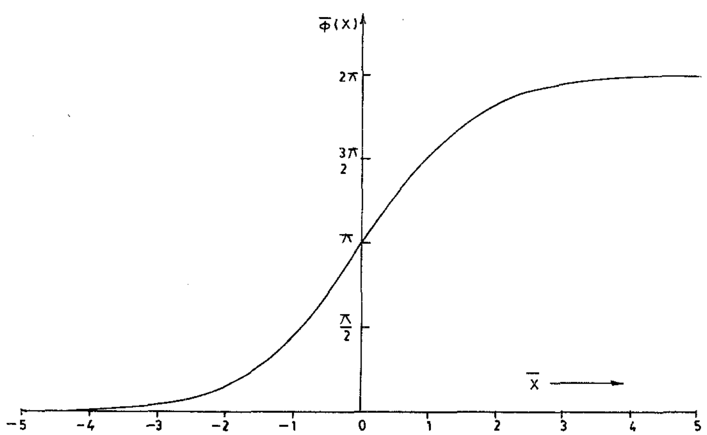

Fig. 1

In deriving (13), we have used

$$V(\varphi)=V(\hat{\varphi})+\bar{\varphi}V^{(1)}(\hat{\varphi})+\frac{\bar{\varphi}^2}{2!}V^{(2)}(\hat{\varphi})+\cdots \tag{14}$$

and

$$\langle\bar{\varphi}^{2n}\rangle=\frac{(2n)!}{2^n n!}\langle\bar{\varphi}^2\rangle^n \tag{15}$$

The expectation value of $\cos[(\sqrt{\lambda}/\mu)\varphi]$ can be directly obtained by writing

$$\cos\left(\frac{\sqrt{\lambda}}{\mu}\varphi\right)=\frac{1}{2}\left[\exp\left(-i\frac{\sqrt{\lambda}}{\mu}\varphi\right)+\exp\left(i\frac{\sqrt{\lambda}}{\mu}\varphi\right)\right]$$

and then performing a functional integration.

The effective potential now reduces to

$$\Gamma=\int dt\left\{\int_x [\hat{\pi}\dot{\varphi}-\tfrac{1}{2}\hat{\pi}^2-\tfrac{1}{2}(\nabla\varphi)^2-\langle V(\varphi)\rangle]\right.$$
$$\left.+\hbar\left[\int_{x,y}\Sigma\dot{G}-2\int_{x,y,z}\Sigma G\Sigma-\int_x(\tfrac{1}{8}G^{-1}-\tfrac{1}{2}\nabla_x^2 G)\right]\right\} \tag{16}$$

The variational equations are

$$\text{(i)}\quad \frac{\delta\Gamma}{\delta\hat{\varphi}}=0\rightarrow\dot{\hat{\pi}}(x,t)=\nabla^2\varphi(x,t)-\frac{\delta}{\delta\hat{\varphi}}\langle V\rangle \tag{17}$$

$$\text{(ii)}\quad \frac{\delta\Gamma}{\delta\hat{\pi}}=0\rightarrow\hat{\pi}(x,t)=\dot{\hat{\varphi}}(x,t) \tag{18}$$

$$\text{(iii)}\quad \frac{\delta\Gamma}{\delta\Sigma}=0\rightarrow\dot{G}(x,y,t)$$
$$=2\left\{\int_z[\Sigma(x,y,t)G(z,y,t)+G(x,z,t)\Sigma(x,y,t)]\right\} \tag{19}$$

$$\text{(iv)} \quad \frac{\delta\Gamma}{\delta G} = 0 \to \dot{\Sigma}(x, y, t) + 2 \int_z \Sigma(x, z, t)\Sigma(z, y, t)$$

$$= \frac{1}{8} G^{-2}(x, y, t) + \frac{1}{2}\left(-\frac{\delta\langle V\rangle}{\delta G}\right)\delta(x-y) \tag{20}$$

Henceforth we shall put $\hbar = 1$.

3. RENORMALIZATION OF THE EFFECTIVE POTENTIAL

We first consider the static case. Here the effective action Γ reduces to an effective potential V_{eff} and is given by

$$\Gamma = -\int dt\, dx\, V_{\text{eff}} \tag{21}$$

We further take $\hat{\varphi}(x, t) = \text{const}$ (i.e., stationary static case).

Then V_{eff} takes the form

$$V_{\text{eff}} = \frac{\mu^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{\mu}\varphi\right)\exp\left(-\frac{\lambda G}{2\mu^2}\right)\right] + \frac{1}{8}G^{-1} - \frac{1}{2}\nabla_x^2 G \tag{22}$$

or

$$V_{\text{eff}} = \frac{G^{-1}}{4} + \frac{\mu^4}{\lambda} - \frac{\mu^2}{2}\cos\left(\frac{\sqrt{\lambda}}{\mu}\hat{\varphi}\right)\exp\left(-\frac{\lambda G}{2\mu^2}\right)\left(G + \frac{2\mu^2}{\lambda}\right) \tag{23}$$

In deriving (23) we have used equation (20) and the result

$$\frac{\delta\langle V\rangle}{\delta G} = \frac{\mu^2}{2}\cos\left(\frac{\sqrt{\lambda}}{\mu}\hat{\varphi}\right)\exp\left(-\frac{\lambda G}{2\mu^2}\right) \tag{24}$$

G satisfies the following gap equation:

$$\frac{1}{4}G^{-2}(x, x') = \left[-\nabla_x^2 + \mu^2\cos\left(\frac{\sqrt{\lambda}}{\mu}\hat{\varphi}\right)\exp\left(-\frac{\lambda G(x, x)}{2\mu^2}\right)\right]\delta(x - x') \tag{25}$$

Next we introduce a mass m defined by

$$m^2 = \mu^2\cos\left(\frac{\sqrt{\lambda}}{\mu}\varphi\right)\exp\left[-\frac{\lambda G(x, x)}{2\mu^2}\right] \tag{26}$$

Using (26), we get from (25)

$$G(x, x)=\int_k \frac{1}{2(k^2+m^2)^{1/2}} \tag{27}$$

where

$$\int_k \equiv \int \frac{dk}{2\pi} \tag{28}$$

μ^2 and λ are bare quantities which require renormalization. The renormalized parameters are defined by

$$m_R^2=\left.\frac{\delta^2 V_{\rm eff}}{\delta \hat{\varphi}^2}\right|_{\hat{\varphi}=0}=\mu^2 \exp\left[-\frac{\lambda_1 G(m_R)}{2}\right] \tag{29}$$

and

$$\lambda_R=-\left.\frac{\delta^4 V_{\rm eff}}{\delta \hat{\varphi}^4}\right|_{\hat{\varphi}=0}=\lambda_1 m_R^2\frac{(1+\lambda_1/4\pi)}{(1-\lambda_1/8\pi)} \tag{30}$$

where

$$\lambda_1=\frac{\lambda}{\mu^2} \tag{31}$$

Equation (30) shows that λ_1 is finite, implying only mass renormalization is sufficient in $(1+1)$ dimensions.

To express $V_{\rm eff}$ in terms of the renormalized parameters, we use the following results, which can be obtained by explicit integration:

$$G(m)-G(m_R)=-\frac{1}{4\pi}\ln\frac{m^2}{m_R^2} \tag{32}$$

and

$$\tfrac{1}{4}[G^{-1}(m)-G^{-1}(m_R)]=\tfrac{1}{2}(m^2-m_R^2)G(m_R) - \frac{m_R^2}{8\pi}\left(\frac{m^2}{m_R^2}\ln\frac{m^2}{m_R^2}-\frac{m^2}{m_R^2}+1\right) \tag{33}$$

These results are similar to those obtained for I_0 and I_1 by Stevenson (1985).

After subtracting the zero-point energy, the resulting renormalized effective potential is found to be

$$V_{\rm eff}=(m^2-m_R^2)\left(\frac{1}{8\pi}-\frac{1}{\lambda_1}\right) \tag{34}$$

where

$$\left(\frac{m^2}{m_R^2}\right)^{1-\lambda_1/8\pi} = m_R^2 \cos(\sqrt{\lambda_1}\ \hat{\varphi}) \tag{35}$$

These results agree with those obtained by Roy *et al.* (1989) using the GEP approach. However, the GEP method, as it stands, does not give any insight into the time evolution of the system. In the following we proceed to investigate the time evolution of the system, starting with the free theory solution.

4. FREE THEORY SOLUTION

The variation equations (with $\hat{\varphi}=0$) are

$$\dot{G}(k,t) = 4\Sigma(k,t)G(k,t) \tag{36}$$

$$\dot{\Sigma}(k,t) = \frac{1}{8}G^{-2}(k,t) - 2\Sigma^2(k,t) - \frac{1}{2}\left[k^2 + \mu^2 \exp\left(-\frac{\lambda_1 G}{2}\right)\right] \tag{37}$$

In the limit $\lambda \to 0$, the above two equations reduce to a single equation,

$$Q = \frac{1}{4Q^3} - w_k^2 Q \tag{38}$$

where

$$Q^2 \equiv G_0 \qquad \text{and} \qquad w_k \equiv (k^2+\mu^2)^{1/2} \tag{39}$$

After some straightforward mathematics, we derive the following expression for $G_0(k, t)$:

$$G_0(k,t) = \frac{1}{2w_k}\{1 + 2n_k - [(1+2n_k)^2 - 1]^{1/2} \cos 2[w_k t - \delta_0(k)]\} \tag{40}$$

with the phase given by

$$\cot 2\delta_0(k) = \frac{w_k}{G(k,0)\dot{G}(k,0)}\left[G^2(k,0) - \frac{\dot{G}^2(k,0)+1}{4w_k^2}\right] \tag{41}$$

and the average energy of the kth mode being

$$E_k = \left(n_k + \frac{1}{2}\right) w_k = \frac{\dot{G}^2(k,0)}{8G(k,0)} + \frac{1}{8}G^{-1}(k,0) + \frac{1}{2}w_k^2 G(k,0) \tag{42}$$

n_k is the average particle number of the kth mode. From (42)

$$n_k = \frac{1}{8w_k G}\left[(1-2w_k G)^2 + \dot{G}^2\right] \tag{43}$$

The initial states should be so chosen that the average particle number density remains finite. In other words, n_k must satisfy

$$\int_k n_k < \infty \tag{44}$$

implying

$$n_k \underset{k\to\infty}{\sim} \frac{1}{k^2} \tag{45}$$

From equation (43), the initial states then have $G(k, 0)$ and $\dot{G}(k, 0)$ of the form

$$G(k, 0) = \frac{1}{2(k^2+\bar{m}^2)^{1/2}}\left[1+f(k)\right] \tag{46}$$

with

$$\operatorname*{Lim}_{k\to\infty} f(k) = \frac{1}{k}\left[a + b\cos\alpha(k)\right] \tag{47}$$

and

$$\operatorname*{Lim}_{k\to\infty} \dot{G}(k, 0) = \frac{1}{k}\left[A + B\cos\beta(k)\right] \tag{48}$$

Here a, b, A, and B are k-independent constants and $\bar{m}$ is a mass parameter (in fact $\bar{m}$ denotes the initial mass). α and β are nonoscillatory.

5. TIME EVOLUTION OF THE SOLITON MASS

We define a time-dependent mass term

$$m^2(t) = \mu^2 \cos(\sqrt{\lambda_1}\,\bar{\varphi}) \exp\left[-\frac{\lambda_1}{2}\int_k G(k, t)\right] \tag{49}$$

We consider the simple case $\bar{\varphi}=0$. However, renormalizability is unaffected by the value of $\bar{\varphi}$. Thus,

$$m^2(t) = m_R^2 \exp[-\tilde{m}^2(t)] \tag{50}$$

where

$$\tilde{m}^2(t) = \frac{\lambda_1}{2} \int_k [G(k, t) - G_v(k)] \tag{51}$$

and

$$G_v(k) = \frac{1}{2(k^2 + m_R^2)^{1/2}} \tag{52}$$

The time-dependent equation for G is therefore

$$\begin{aligned}\ddot{G}(k, t) = {} & \tfrac{1}{2} G^{-1}(k, t) + \tfrac{1}{2} G^{-1}(k, t) \dot{G}^2(k, t) \\ & - 2\{k^2 + m_R^2 \exp[-\tilde{m}^2(t)]\} G(k, t)\end{aligned} \tag{53}$$

The $k \to \infty$ limit of this equation is identical to the asymptotic limit of the free equation without the mass terms.

Our next objective is to find the next to leading order terms which depend on the time-dependent mass. For this, we can apply the perturbation method described in detail by Pi and Samiullah (1987). However, note that to show the finiteness of $m^2(t)$, we keep terms up to $O(\lambda_1)$ only, as the terms containing higher order in λ_1 will converge for large k if the terms of the first order do so. For the sake of brevity we give only the outline of the calculation, as the entire calculation is rather lengthy and tedious.

For our perturbation calculations, we define a quantity

$$\Omega(k, t) = \tfrac{1}{2} G^{-1}(k, t) - 2i\Sigma(k, t) \tag{54}$$

in terms of which the variational equations (36) and (37) reduce to a single equation

$$\frac{i\,\partial\Omega}{\partial t} = \Omega^2 - [k^2 + m^2(t)] \tag{55}$$

Next we expand $\tilde{m}^2(t)$ and $\Omega(k, t)$ as

$$\tilde{m}^2(t) = \sum_{n=1} \lambda_1^n \tilde{m}_n^2(t) \tag{56}$$

$$\Omega(k, t) = \sum_{n=0} \lambda_1^n \Omega_n(k, t) \tag{57}$$

and retain terms only up to order λ_1 in (50).

We skip the details for reasons mentioned earlier and quote the final results only:

$$G(k, t) \underset{k \to \infty}{\sim} \frac{1}{2k} - \frac{m^2(t)}{4k^3} + \frac{1}{2k^2}[a + b \cos \alpha(k)] \cos 2kt$$

$$+ \frac{1}{2k^2}[A + B \cos \beta(k)] \sin 2kt + O\left(\frac{1}{k^3}\right) \tag{58}$$

From (51) and (58), we get

$$\tilde{m}^2(t) = \frac{\lambda_1}{2} \int_k \left[\frac{a + b \cos \alpha(k)}{2k^2} \cos 2kt \right.$$

$$\left. + \frac{A + B \cos \beta(k)}{2k^2} \sin 2kt \right] + O\left(\frac{1}{k^3}\right) \tag{59}$$

In deriving (58) and (59) we have chosen the initial mass $\bar{m} = \tilde{m}(0)$ so that $f(k)$ determines the excitation of $G(k, 0)$ relative to $G_v(k)$.

It is clear from (59) that $\tilde{m}^2(t)$ is finite for all t. Hence, from (50) and (51) it is clear that the soliton mass $m^2(t)$ remains finite for all t.

6. SUMMARY AND REMARKS

We have applied a time-dependent variational method to sine-Gordon theory in $1 + 1$ dimensions. As mentioned earlier, sine-Gordon theory has a static localized soliton solution. The time-dependent variational method enables us to find the time evolution of the system, particularly the time dependence of the mass. Our static solution reproduces the GEP result obtained in Roy *et al.* (1989). This is not surprising, as the GEP is also essentially a scaling variational method, but unfortunately there is no procedure for investigating the time evolution in the GEP method. In this respect the time-dependent scaling variational method is quite useful. We have shown that $m^2(t)$ of the sine-Gordon theory is finite for all t.

REFERENCES

Cooper, F., and Mottola, E. (1987). *Physical Review D*, **36**, 3114.
Cooper, F., Pi, S.-Y., and Stancioff, P. N. (1986). *Physical Review D*, **34**, 3831.
Jackiw, R., and Kerman, A. (1979). *Physical Letters*, **71A**, 158.
Kim, S. K., Namgung, W., Soh, K. S., and Lee, J. H. (1988). *Physical Review D*, **38**, 1853.
Pi, S.-Y., and Samiullah, M. (1987). *Physical Review D*, **36**, 3128.
Rajaraman, R. (1982). *Solitons and Instantons in Quantum Field Theory*, North-Holland, Amsterdam.

Roy, B. (1991). *International Journal of Modern Physics A*, **6**, 1525.
Roy, P., Roychoudhury, R., and Varshni, Y. P. (1989). *Modern Physics Letters A*, **4**, 2031.
Stevenson, P. M. (1985). *Physical Review D*, **32**, 1389.